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ON SOME MEAN SQUARE ESTIMATES IN THE RANKIN-SELBERG PROBLEM

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An overview of the classical Rankin-Selberg problem involving the asymptotic formula for sums of coefficients of holomorphic cusp forms is given. We also study the function $\Delta(x; \xi)$ ($0 \leq \xi \leq 1$), the error term in the Rankin-Selberg problem weighted by ξ -th power of the logarithm. Mean square estimates for $\Delta(x; \xi)$ are proved.

1. THE RANKIN-SELBERG PROBLEM

The classical Rankin-Selberg problem consists of the estimation of the error term function

$$(1.1) \quad \Delta(x) := \sum_{n \leq x} c_n - Cx,$$

where the notation is as follows. Let $\varphi(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $SL(2, \mathbb{Z})$, and denote by $a(n)$ the n -th Fourier coefficient of $\varphi(z)$ (see e.g., R.A. Rankin [15] for a comprehensive account). We suppose that $\varphi(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$, that is, $a(1) = 1$ and $T(n)\varphi = a(n)\varphi$ for every $n \in \mathbb{N}$. In (1.1) $C > 0$ is a suitable constant (see e.g., [9] for its explicit expression), and c_n is the convolution function defined by

$$c_n = n^{1-\kappa} \sum_{m^2 | n} m^{2(\kappa-1)} \left| a\left(\frac{n}{m^2}\right) \right|^2.$$

The classical Rankin-Selberg bound of 1939 is

$$(1.2) \quad \Delta(x) = O(x^{3/5}),$$

hitherto unimproved. In their works, done independently, R.A. Rankin [16] derives (1.2) from a general result of E. Landau [11], while A. Selberg [17] states the result

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with no proof. Although the exponent $3/5$ in (1.2) represents one of the longest standing records in analytic number theory, recently there have been some developments in some other aspects of the Rankin-Selberg problem. In this paper we shall present an overview of some of these new results. In addition, we shall consider the weighted sum (the so-called Riesz logarithmic means of order ξ), namely

$$(1.3) \quad \frac{1}{\Gamma(\xi+1)} \sum_{n \leq x} c_n \log^\xi \left(\frac{x}{n} \right) := Cx + \Delta(x; \xi) \quad (\xi \geq 0),$$

where C is as in (1.1), so that $\Delta(x) \equiv \Delta(x; 0)$. The effect of introducing weights such as the logarithmic weight in (1.3) is that the ensuing error term (in our case this is $\Delta(x; \xi)$) can be estimated better than the original error term (i.e., in our case $\Delta(x; 0)$). This was shown by Matsumoto, Tanigawa and the author in [9], where it was proved that

$$(1.4) \quad \Delta(x; \xi) \ll_\varepsilon x^{(3-2\xi)/5+\varepsilon} \quad (0 \leq \xi \leq 3/2).$$

Here and later ε denotes arbitrarily small constants, not necessarily the same ones at each occurrence, while $a \ll_\varepsilon b$ means that the constant implied by the \ll -symbol depends on ε . When $\xi = 0$ we recover (1.2) from (1.4), only with the extra ‘ ε ’ factor present. In this work we shall pursue the investigations concerning $\Delta(x; \xi)$, and deal with mean square bounds for this function.

2. THE FUNCTIONAL EQUATIONS

In view of (1.1) and (1.2) it follows that the generating Dirichlet series

$$(2.1) \quad Z(s) := \sum_{n=1}^{\infty} c_n n^{-s} \quad (s = \sigma + it)$$

converges absolutely for $\sigma > 1$. The arithmetic function c_n is multiplicative and satisfies $c_n \ll_\varepsilon n^\varepsilon$. Moreover, it is well known (see e.g., R.A. Rankin [14], [15]) that $Z(s)$ satisfies for all s the functional equation

$$(2.2) \quad \Gamma(s + \kappa - 1) \Gamma(s) Z(s) = (2\pi)^{4s-2} \Gamma(\kappa - s) \Gamma(1 - s) Z(1 - s),$$

which provides then the analytic continuation of $Z(s)$. In modern terminology $Z(s)$ belongs to the Selberg class \mathcal{S} of L -functions of degree four (see A. Selberg [18] and the survey paper of Kaczorowski–Perelli [10]). An important feature, proved by G. Shimura [19] (see also A. Sankaranarayanan [16]) is

$$(2.3) \quad Z(s) = \zeta(s) \sum_{n=1}^{\infty} b_n n^{-s} = \zeta(s) B(s),$$

where $B(s)$ is holomorphic for $\sigma > 0$, $b_n \ll_\varepsilon n^\varepsilon$ (in fact $\sum_{n \leq x} b_n^2 \leq x \log^A x$ holds, too). It also satisfies the functional equation

$$\begin{aligned} B(s)\Delta_1(s) &= B(1-s)\Delta_1(1-s), \\ \Delta_1(s) &= \pi^{-3s/2}\Gamma(\tfrac{1}{2}(s+\kappa-1))\Gamma(\tfrac{1}{2}(s+\kappa))\Gamma(\tfrac{1}{2}(s+\kappa+1)), \end{aligned}$$

and actually $B(s) \in \mathcal{S}$ with degree three. The decomposition (2.3) (the so-called ‘Shimura lift’) allows one to use, at least to some extent, results from the theory of $\zeta(s)$ in connection with $Z(s)$, and hence to derive results on $\Delta(x)$.

3. THE COMPLEX INTEGRATION APPROACH

A natural approach to the estimation of $\Delta(x)$, used by the author in [8], is to apply the classical complex integration technique. We shall briefly present this approach now. On using Perron’s inversion formula (see e.g., the Appendix of [3]), the residue theorem and the convexity bound $Z(s) \ll_\varepsilon |t|^{2-2\sigma+\varepsilon}$ ($0 \leq \sigma \leq 1$, $|t| \geq 1$), it follows that

$$(3.1) \quad \Delta(x) = \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{Z(s)}{s} x^s ds + O_\varepsilon \left(x^\varepsilon \left(x^{1/2} + \frac{x}{T} \right) \right) \quad (1 \ll T \ll x).$$

If we suppose that

$$(3.2) \quad \int_X^{2X} |B(\tfrac{1}{2} + it)|^2 dt \ll_\varepsilon X^{\theta+\varepsilon} \quad (\theta \geq 1),$$

and use the elementary fact (see [3] for the results on the moments of $|\zeta(\frac{1}{2} + it)|$) that

$$(3.3) \quad \int_X^{2X} |\zeta(\tfrac{1}{2} + it)|^2 dt \ll X \log X,$$

then from (2.3), (3.2), (3.3) and the Cauchy-Schwarz inequality for integrals we obtain

$$\int_X^{2X} |Z(\tfrac{1}{2} + it)| dt \ll_\varepsilon X^{(1+\theta)/2+\varepsilon}.$$

Therefore (3.1) gives

$$(3.4) \quad \Delta(x) \ll_\varepsilon x^\varepsilon (x^{1/2} T^{\theta/2-1/2} + x T^{-1}) \ll_\varepsilon x^{\frac{\theta}{\theta+1}+\varepsilon}$$

with $T = x^{1/(\theta+1)}$. This was formulated in [8] as

THEOREM A. *If θ is given by (3.2), then*

$$(3.5) \quad \Delta(x) \ll_\varepsilon x^{\frac{\theta}{\theta+1}+\varepsilon}.$$

To obtain a value for θ , note that $B(s)$ belongs to the Selberg class of degree three, hence $B(\frac{1}{2} + it)$ in (3.2) can be written as a sum of two Dirichlet polynomials (e.g., by the reflection principle discussed in [3, Chapter 4]), each of length $\ll X^{3/2}$. Thus by the mean value theorem for Dirichlet polynomials (op. cit.) we have $\theta \leq 3/2$ in (3.2). Hence (3.5) gives (with unimportant ε) the Rankin-Selberg bound $\Delta(x) \ll_\varepsilon x^{3/5+\varepsilon}$. Clearly improvement will come from better values of θ . Note that the best possible value of θ in (3.2) is $\theta = 1$, which follows from general results on Dirichlet series (see e.g., [3, Chapter 9]). It gives $1/2 + \varepsilon$ as the exponent in the Rankin-Selberg problem, which is the limit of the method (the conjectural exponent $3/8 + \varepsilon$, which is best possible, is out of reach; see the author's work [4]). To attain this improvement one faces essentially the same problem as in proving the sixth moment for $|\zeta(\frac{1}{2} + it)|$, namely

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^6 dt \ll_\varepsilon T^{1+\varepsilon},$$

only this problem is even more difficult, because the arithmetic properties of the coefficients b_n are even less known than the properties of the divisor coefficients

$$d_3(n) = \sum_{abc=n; a,b,c \in \mathbb{N}} 1,$$

generated by $\zeta^3(s)$. If we knew the analogue of the strongest sixth moment bound

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^6 dt \ll T^{5/4} \log^C T \quad (C > 0),$$

namely the bound (3.2) with $\theta = 5/4$, then (3.1) would yield $\Delta(x) \ll_\varepsilon x^{5/9+\varepsilon}$, improving substantially (1.2).

The essential difficulty in this problem may be seen indirectly by comparing it with the estimation of $\Delta_4(x)$, the error term in the asymptotic formula for the summatory function of $d_4(n) = \sum_{abcd=n; a,b,c,d \in \mathbb{N}} 1$. The generating function in this case is $\zeta^4(s)$. The problem analogous to the estimation of $\Delta(x)$ is to estimate $\Delta_4(x)$, given the product representation

$$(3.6) \quad \sum_{n=1}^{\infty} d_4(n) n^{-s} = \zeta(s) G(s) = \zeta(s) \sum_{n=1}^{\infty} g(n) n^{-s} \quad (\sigma > 1)$$

with $g(n) \ll_\varepsilon n^\varepsilon$ and $G(s)$ of degree three in the Selberg class (with a pole of order three at $s = 1$). By the complex integration method one gets $\Delta_4(x) \ll_\varepsilon x^{1/2+\varepsilon}$ (here ' ε ' may be replaced by a log-factor) using the classical elementary bound $\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt \ll T \log^4 T$. Curiously, this bound for $\Delta_4(x)$ has never been improved; exponential sum techniques seem to give a poor result here. However, if one knows only (3.6), then the situation is quite analogous to the Rankin-Selberg problem, and nothing better than the exponent $3/5$ seems obtainable. The bound $\Delta(x) \ll_\varepsilon x^{1/2+\varepsilon}$ follows also directly from (3.1) if the Lindelöf hypothesis for $Z(s)$ (that $Z(\frac{1}{2} + it) \ll_\varepsilon |t|^\varepsilon$) is assumed.

4. MEAN SQUARE OF THE RANKIN-SELBERG ZETA-FUNCTION

Let, for a given $\sigma \in \mathbb{R}$,

$$(4.1) \quad \mu(\sigma) = \limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{\log t}$$

denote the Lindelöf function (the famous, hitherto unproved, Lindelöf conjecture for $\zeta(s)$ is that $\mu(\sigma) = 0$ for $\sigma \geq \frac{1}{2}$, or equivalently that $\zeta(\frac{1}{2} + it) \ll_\varepsilon |t|^\varepsilon$). In [8] the author proved the following

THEOREM B. *If $\beta = 2/(5 - \mu(\frac{1}{2}))$, then for fixed σ satisfying $\frac{1}{2} < \sigma \leq 1$ we have*

$$(4.2) \quad \int_1^T |Z(\sigma + it)|^2 dt = T \sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O_\varepsilon(T^{(2-2\sigma)/(1-\beta)+\varepsilon}).$$

This result is the sharpest one yet when σ is close to 1. For σ close to $\frac{1}{2}$ one cannot obtain an asymptotic formula, but only the upper bound (this is [7, eq. (9.27)])

$$(4.3) \quad \int_T^{2T} |Z(\sigma + it)|^2 dt \ll_\varepsilon T^{2\mu(1/2)(1-\sigma)+\varepsilon} (T + T^{3(1-\sigma)}) \quad (\tfrac{1}{2} \leq \sigma \leq 1).$$

The upper bound in (4.3) follows easily from (2.3) and the fact that, as already mentioned, $B(s) \in \mathcal{S}$ with degree three, so that $B(\frac{1}{2} + it)$ can be approximated by Dirichlet polynomials of length $\ll t^{3/2}$, and the mean value theorem for Dirichlet polynomials yields

$$\int_T^{2T} |B(\sigma + it)|^2 dt \ll_\varepsilon T^\varepsilon (T + T^{3(1-\sigma)}) \quad (\tfrac{1}{2} \leq \sigma \leq 1).$$

Note that with the sharpest known result (see M.N. Huxley [2]) $\mu(1/2) \leq 32/205$ we obtain $\beta = 410/961 = 0.426638917\dots$. The limit is the value $\beta = 2/5$ if the Lindelöf hypothesis (that $\mu(\frac{1}{2}) = 0$) is true. Thus (4.2) provides a true asymptotic formula for

$$\sigma > \frac{1+\beta}{2} = \frac{1371}{1922} = 0.7133194\dots$$

The proof of (4.2), given in [8], is based on the general method of the author's paper [6], which contains a historic discussion on the formulas for the left-hand side of (4.2) (see also K. Matsumoto [12]).

We are able to improve (4.2) in the case when $\sigma = 1$. The result is contained in

THEOREM 1. *We have*

$$(4.4) \quad \int_1^T |Z(1+it)|^2 dt = T \sum_{n=1}^{\infty} c_n^2 n^{-2} + O_{\varepsilon}((\log T)^{2+\varepsilon}).$$

Proof. For $\sigma = \Re s > 1$ and $X \geq 2$ we have

(4.5)

$$\begin{aligned} Z(s) &= \sum_{n \leq X} c_n n^{-s} + \int_X^{\infty} x^{-s} d\left(\sum_{n \leq x} c_n\right) \\ &= \sum_{n \leq X} c_n n^{-s} + \frac{CX^{1-s}}{s-1} - \Delta(x)X^{-s} - s \int_X^{\infty} \Delta(x)x^{-s-1} dx. \end{aligned}$$

By using (1.2) it is seen that the last integral converges absolutely for $\sigma = \Re s > 3/5$, so that (4.5) provides the analytic continuation of $Z(s)$ to this region. Taking $s = 1 + it$, $1 \leq t \leq T$, $X = T^{10}$, it follows that

$$(4.6) \quad \int_1^T |Z(1+it)|^2 dt = \int_1^T \left\{ \left| \sum_{n \leq X} c_n n^{-1-it} \right|^2 - 2C \Im \left(\sum_{n \leq X} \frac{c_n}{nt} \left(\frac{X}{n} \right)^{it} \right) \right\} dt + O(1).$$

By the mean value theorem for Dirichlet polynomials we have

$$\begin{aligned} \int_1^T \left| \sum_{n \leq X} c_n n^{-1-it} \right|^2 dt &= T \sum_{n \leq X} c_n^2 + O\left(\sum_{n \leq X} c_n^2 n^{-1}\right) \\ &= T \sum_{n=1}^{\infty} c_n^2 + O_{\varepsilon}((\log T)^{2+\varepsilon}), \end{aligned}$$

where we used the bound (see K. Matsumoto [12])

$$(4.7) \quad \sum_{n \leq x} c_n^2 \ll_{\varepsilon} x(\log x)^{1+\varepsilon}$$

and partial summation. Finally we have

$$(4.8) \quad \sum_{n \leq X} \frac{c_n}{n} \int_1^T \frac{1}{t} \left(\frac{X}{n} \right)^{it} dt \ll \log \log T.$$

To see that (4.8) holds, note first that for $X - X/\log T \leq n \leq X$ the integral over t is trivially estimated as $\ll \log T$, and the total contribution of such n is

$$\ll \log T \sum_{X-X/\log T \leq n \leq X} \frac{c_n}{n} dx \ll 1$$

on using (1.1)–(1.2). For the remaining n we note that the integral over t equals

$$\frac{\left(\frac{X}{n}\right)^{it}}{it \log(X/n)} \Big|_1^T + \frac{1}{i \log(X/n)} \int_1^T \left(\frac{X}{n}\right)^{it} \frac{dt}{t^2}.$$

The contribution of those n is, using (1.1)–(1.2) again and making the change of variable $X/u = v$,

$$\begin{aligned} &\ll \sum_{1 \leq n \leq X - X/\log T} \frac{c_n}{n \log(X/n)} = \int_{1-0}^{X-X/\log T} \frac{1}{u \log(X/u)} d(Cu + \Delta(u)) \\ &= \int_1^{X-X/\log T} \frac{1}{u \log(X/u)} \left(C + \frac{\Delta(u)}{u} + \frac{\Delta(u)}{u \log(X/u)} \right) du + O(1) \\ &\ll \int_1^{X-X/\log T} \frac{du}{u \log(X/u)} + 1 = \int_{(1-1/\log T)^{-1}}^X \frac{dv}{v \log v} + 1 \\ &= \log \log X - \log \log(1 - 1/\log T)^{-1} + 1 \ll \log \log T, \end{aligned}$$

and (4.8) follows.

One can improve the error term in (4.4) to $O(\log^2 T)$, which is the limit of the method. I am very grateful to Prof. Alberto Perelli, who has kindly indicated this to me. The argument is very briefly as follows. Note that the coefficients c_n^2 are essentially the tensor product of the c_n 's, and the c_n are essentially the tensor product of the $a(n)$'s; “essentially” means in this case that the corresponding L -functions differ at most by a “fudge factor”, i.e., a Dirichlet series converging absolutely for $\sigma > 1/2$ and non-vanishing at $s = 1$. In terms of L -functions, the tensor product of the $a(n)$ (the coefficients of the tensor square L -function) corresponds to the product of $\zeta(s)$ and the L -function of Sym^2 (Shimura's lift). Moreover, Gelbart–Jacquet [1] have shown that Sym^2 is a cuspidal automorphic representation, so one can apply to the above product the general Rankin-Selberg theory to obtain “good properties” of the corresponding L -function. Since Sym^2 is irreducible, the L -function corresponding to c_n^2 has a double pole at $s = 1$ and a functional equation of Riemann type. It follows that the sum in (4.7) is asymptotic to $Dx \log x$ for some $D > 0$, and the assertion follows by following the preceding argument.

In concluding this section, let it be mentioned that, using (4.5), it easily follows that $Z(1+it) \ll \log |t|$ ($t \geq 2$).

5. MEAN SQUARE OF $\Delta(x; \xi)$

In this section we shall consider mean square estimates for $\Delta(x; \xi)$, defined by (1.3). Although we could consider the range $\xi > 1$ as well, for technical reasons we shall restrict ourselves to the range $0 \leq \xi \leq 1$, which is the condition that will be assumed henceforth to hold. Let

$$(5.1) \quad \beta_\xi := \inf \left\{ \beta \geq 0 : \int_1^X \Delta^2(x; \xi) dx \ll X^{1+2\beta} \right\}.$$

The definition of β_ξ is the natural analogue of the classical constants in mean square estimates for the generalized Dirichlet divisor problem (see [3, Chapter 13]). Our first result in this direction is

THEOREM 2. *We have*

$$(5.2) \quad \frac{3-2\xi}{8} \leq \beta_\xi \leq \max\left(\frac{1-\xi}{2}, \frac{3-2\xi}{8}\right) \quad (0 \leq \xi \leq 1).$$

Proof. First of all, note that (5.2) implies that $\beta_\xi = (3-2\xi)/8$ for $\frac{1}{2} \leq \xi \leq 1$, so that in this interval the precise value of β_ξ is determined. The main tool in our investigations is the explicit Voronoï type formula for $\Delta(x; \xi)$. This is

$$(5.3) \quad \Delta(x; \xi) = V_\xi(x, N) + R_\xi(x, N),$$

where, for $N \gg 1$,

$$(5.4)$$

$$\begin{aligned} V_\xi(x, N) &= (2\pi)^{-1-\xi} x^{(3-2\xi)/8} \sum_{n \leq N} c_n n^{-(5+2\xi)/8} \cos\left(8\pi(xn)^{1/4} + \frac{1}{2}(\frac{1}{2} - \xi)\pi\right), \\ R_\xi(x, N) &\ll_\varepsilon (xN)^\varepsilon \left(1 + x^{(3-\xi)/4} N^{-(1+\xi)/4} + (xN)^{(1-\xi)/4} + x^{(1-2\xi)/8}\right). \end{aligned}$$

This follows from the work of U. Vorhauer [20] (for $\xi = 0$ this is also proved in [9]), specialized to the case when

$$A = \frac{1}{(2\pi)^2}, B = (2\pi)^4, M = L = 2, b_1 = b_2 = d_1 = d_2 = 1, \beta_1 = \kappa - \frac{1}{2}, b_2 = \frac{1}{2},$$

$$\delta_1 = \kappa - \frac{3}{2}, \delta_2 = -\frac{1}{2}, \gamma = 1, p = B, q = 4, \lambda = 2, \Lambda = -1, C = (2\pi)^{-5/2}.$$

In (5.3)–(5.4) we take $N = x$, so that $R_\xi(x, N) \ll_\varepsilon x^{(1-\xi)/2+\varepsilon}$. Since $\frac{1-\xi}{2} \leq \frac{3-2\xi}{8}$ for $\xi \geq \frac{1}{2}$, the lower bound in (5.2) follows by the method of [4]. For the upper bound we use $c_n \ll_\varepsilon n^\varepsilon$ and note that $(e(z) = \exp(2\pi iz))$

$$\begin{aligned} &\left| \int_X^{2X} \sum_{K < k \leq 2K} c_k k^{-(5+2\xi)/8} e(4(xk)^{1/4}) \right|^2 dx \\ &\ll X + \sum_{k_1 \neq k_2} c_{k_1} c_{k_2} (k_1 k_2)^{-(5+2\xi)/8} \int_X^{2X} e(4x^{1/4}(k_1^{1/4} - k_2^{1/4})) dx \\ &\ll_\varepsilon X + X^{3/4+\varepsilon} K^{-(5+2\xi)/4} \sum_{k_1 \neq k_2} \left| k_1^{1/4} - k_2^{1/4} \right|^{-1} \\ &\ll_\varepsilon X + X^{3/4+\varepsilon} K^{(1-\xi)/2}, \end{aligned}$$

where we used the first derivative test (cf. [3, Lemma 2.1]). Since $K \ll X$ and

$$\int_X^{2X} \Delta^2(x; \xi) dx \ll \int_X^{2X} |V_\xi(x, N)|^2 dx + \int_X^{2X} |R(x, N)|^2 dx,$$

it follows that

$$\int_X^{2X} \Delta^2(x; \xi) dx \ll_\varepsilon X^{(7-2\xi)/4+\varepsilon} + X^{2-\xi+\varepsilon},$$

which clearly proves the assertion.

Our last result is a bound for β_ξ , which improves on (5.2) when ξ is small. This is

THEOREM 3. *We have*

$$(5.5) \quad \beta_\xi \leq \frac{2-2\xi}{5-2\mu(\frac{1}{2})} \quad \left(0 \leq \xi \leq \frac{1}{6}(1+2\mu(\frac{1}{2})) \right).$$

Proof. We start from

$$(5.6) \quad \Delta(x; \xi) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT} Z(s) \frac{x^s}{s^{\xi+1}} ds,$$

where $0 < c = c(\xi) < 1$ is a suitable constant (see K. Matsumoto [13] for a detailed derivation of formulas analogous to (5.6)). By the Mellin inversion formula we have (see e.g., the Appendix of [3])

$$Z(s)s^{-\xi-1} = \int_0^\infty \Delta(1/x; \xi) x^{s-1} dx \quad (\Re s = c).$$

Hence by Parseval's formula for Mellin transforms (op. cit.) we obtain, for $\beta_\xi < \sigma < 1$,

(5.7)

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^\infty \frac{|Z(\sigma + it)|^2}{|\sigma + it|^{2\xi+2}} dt = \int_0^\infty \Delta^2(1/x; \xi) x^{2\sigma-1} dx \\ & = \int_0^\infty \Delta^2(x; \xi) x^{-2\sigma-1} dx \gg X^{-2\sigma-1} \int_X^{2X} \Delta^2(x; \xi) dx. \end{aligned}$$

Therefore if the first integral converges for $\sigma = \sigma_0 + \varepsilon$, then (5.7) gives

$$\int_X^{2X} \Delta^2(x; \xi) dx \ll X^{2\sigma+1},$$

namely $\beta_\xi \leq \sigma_0$. The functional equation (2.2) and Stirling's formula in the form

$$|\Gamma(s)| = \sqrt{2\pi} |t|^{\sigma-1/2} e^{-\pi|t|/2} (1 + O(|t|^{-1})) \quad (|t| \geq t_0 > 0)$$

imply that

$$(5.8) \quad Z(s) = \mathcal{X}(s)Z(1-s), \quad \mathcal{X}(\sigma+it) \asymp |t|^{2-4\sigma} \quad (s = \sigma+it, 0 \leq \sigma \leq 1, |t| \geq 2).$$

Thus it follows on using (4.3) that

$$\begin{aligned} \int_T^{2T} |Z(\sigma+it)|^2 dt &\ll T^{4-8\sigma} \int_T^{2T} |Z(1-\sigma+it)|^2 dt \\ &\ll_{\varepsilon} T^{4-8\sigma+2\mu(\frac{1}{2})\sigma+\max(1,3\sigma)+\varepsilon}. \end{aligned}$$

But we have

$$4 - 8\sigma + 2\mu(\tfrac{1}{2})\sigma + \max(1, 3\sigma) = 4 - 5\sigma + 2\mu(\tfrac{1}{2})\sigma < 2\xi + 2$$

for

$$(5.9) \quad \sigma > \sigma_0 = \frac{2 - 2\xi}{5 - 2\mu(\frac{1}{2})},$$

provided that $\sigma_0 \geq 1/3$, which occurs if $0 \leq \xi \leq \frac{1}{6}(1 + 2\mu(\frac{1}{2}))$. Thus the first integral in (5.7) converges if (5.9) holds, and Theorem 3 is proved. Note that this result is a generalization of Theorem 7 in [8], which says that $\beta_0 \leq (2 - 2\xi)/(5 - 2\mu(\frac{1}{2}))$.

In the case when $\beta_\xi = (3 - 2\xi)/8$ we could actually derive an asymptotic formula for the integral of the mean square of $\Delta(x; \xi)$, much in the same way that this was done in [9] for the square of

$$\Delta_1(x) := \int_0^x \Delta(u) du,$$

where it was shown that

$$(5.10) \quad \int_1^X \Delta_1^2(x) dx = DX^{13/4} + O_\varepsilon(X^{3+\varepsilon})$$

with explicit $D > 0$ (in [12] the error term was improved to $O_\varepsilon(X^3(\log X)^{3+\varepsilon})$). In the case of $\Delta(x; 1)$ the formula (5.10) may be used directly, since

$$(5.11) \quad \frac{1}{x} \Delta_1(x) = \frac{1}{x} \int_0^x \Delta(u) du = \Delta(x; 1) + O_\varepsilon(x^\varepsilon).$$

To see that (5.11) holds, note that with $c = 1 - \varepsilon$ we have

$$\begin{aligned} \Delta(x; 1) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(s) \frac{x^s}{s^2} ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(s) \frac{x^s}{s(s+1)} ds + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(s) \frac{x^s}{s^2(s+1)} ds \\ &= \frac{1}{x} \int_0^x \Delta(u) du + \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} Z(s) \frac{x^s}{s^2(s+1)} ds \\ &= \frac{1}{x} \Delta_1(x) + O_\varepsilon(x^\varepsilon), \end{aligned}$$

on applying (5.8) to the last integral above.

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